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THE HARDY–LITTLEWOOD MAXIMAL FUNCTION OF A SOBOLEV FUNCTION

BY

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ABSTRACT

We prove that the Hardy–Littlewood maximal operator is bounded in the Sobolev space $W^{1,p}(\mathbf{R}^n)$ for $1 < p \leq \infty$. As an application we study a weak type inequality for the Sobolev capacity. We also prove that the Hardy–Littlewood maximal function of a Sobolev function is quasi-continuous.

1. Introduction

The Hardy–Littlewood maximal function $\mathcal{M}f\colon \mathbf{R}^n \to [0,\infty]$ of a locally integrable function $f\colon \mathbf{R}^n \to [-\infty,\infty]$ is defined by

$$\mathcal{M}f(x) = \sup \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy, \tag{1.1}$$

where the supremum is over all radii $r > 0$. Here $|B(x,r)|$ denotes the volume of the ball $B(x,r)$. The maximal function is a classical tool in harmonic analysis but recently it has been successfully used in studying Sobolev functions and partial differential equations, see [1] and [4]. The celebrated theorem of Hardy, Littlewood and Wiener asserts that the maximal operator is bounded in $L^p(\mathbf{R}^n)$ for $1 < p \leq \infty$,

$$\|\mathcal{M}f\|_p \leq A_p \|f\|_p, \tag{1.2}$$

where the constant A_p depends only on p and n, see [5, Theorem I.1]. This theorem is one of the cornerstones of harmonic analysis but the applications

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to Sobolev functions and to partial differential equations indicate that it would also be useful to know how the maximal operator preserves the differentiability properties of functions. It is easy to show that maximal function of a Lipschitz function is again Lipschitz and hence, in that case, by Rademacher's theorem it is differentiable almost everywhere. Unfortunately, the maximal function of a differentiable function is not differentiable in general. The reason for this is twofold. First, the modulus of a differentiable function is not differentiable and, even though the function would not change signs, the supremum of differentiable functions may fail to be differentiable. The purpose of this note is to show that, however, certain weak differentiability properties are preserved under the maximal operator. Our main theorem is that the Hardy–Littlewood maximal operator is bounded in the Sobolev space $W^{1,p}(\mathbf{R}^n)$ for $1 < p \leq \infty$ and hence, in that case, it has classical partial derivatives almost everywhere. The corresponding result for $p = 1$ fails because then we don't have the Hardy–Littlewood–Wiener theorem available.

Recall, that the Sobolev space $W^{1,p}(\mathbf{R}^n)$, $1 \leq p \leq \infty$ consists of functions $u \in L^p(\mathbf{R}^n)$, whose first weak partial derivatives $D_i u$, $i = 1, 2, \dots, n$, belong to $L^p(\mathbf{R}^n)$. We endow $W^{1,p}(\mathbf{R}^n)$ with the norm

$$\|u\|_{1,p} = \|u\|_p + \|Du\|_p, \tag{1.3}$$

where $Du = (D_1 u, D_2 u, \dots, D_n u)$ is the weak gradient of u. For the basic properties of Sobolev functions we refer to [3, Chapter 7]. Now we are ready to formulate our main result.

1.4. Theorem: *Let $1 < p < \infty$. If $u \in W^{1,p}(\mathbf{R}^n)$, then $\mathcal{M}u \in W^{1,p}(\mathbf{R}^n)$ and*

$$\left|D_i \mathcal{M}u\right| \leq \mathcal{M}D_i u, \quad i = 1, 2, \dots, n, \tag{1.5}$$

almost everywhere in $\mathbf{R}^n$.

2. The proof of Theorem 1.4

If $\chi_{B(0,r)}$ is the characteristic function of $B(0,r)$ and

$$\chi_r = \frac{\chi_{B(0,r)}}{|B(0,r)|},$$

then

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y)|\, dy = |u| * \chi_r(x),$$

where $*$ denotes the convolution. Now $|u| * \chi_r \in W^{1,p}(\mathbf{R}^n)$ and

$$D_i(|u| * \chi_r) = \chi_r * D_i|u|, \quad i = 1, 2, \dots, n,$$

almost everywhere in $\mathbf{R}^n$. Let r_j, $j = 1, 2, \dots$, be an enumeration of positive rationals. Since u is locally integrable, we may restrict ourselves in definition (1.1) to the positive rational radii. Hence

$$\mathcal{M}u(x) = \sup_j (|u| * \chi_{r_j})(x).$$

We define functions $v_k\colon \mathbf{R}^n \to R$, $k = 1, 2, \dots$, by

$$v_k(x) = \max_{1 \le j \le k} (|u| * \chi_{r_j})(x).$$

Now (v_k) is an increasing sequence of functions in $W^{1,p}(\mathbf{R}^n)$ [3, Lemma 7.6] which converges to $\mathcal{M}u$ pointwise and

$$\begin{aligned} |D_i v_k| &\le \max_{1 \le j \le k} \left| D_i(|u| * \chi_{r_j}) \right| \\ &= \max_{1 \le j \le k} \left| \chi_{r_j} * D_i|u| \right| \le \mathcal{M}D_i|u| = \mathcal{M}D_i u, \end{aligned} \tag{2.1}$$

$i = 1, 2, \dots, n$, almost everywhere in $\mathbf{R}^n$. Here we also used the fact that $\left| D_i|u| \right| = |D_i u|$, $i = 1, 2, \dots, n$, almost everywhere. Thus

$$\|Dv_k\|_p \le \sum_{i=1}^n \|D_i v_k\|_p \le \sum_{i=1}^n \|\mathcal{M}D_i u\|_p$$

and the Hardy–Littlewood–Wiener inequality (1.2) implies

$$\begin{aligned} \|v_k\|_{1,p} &\le \|\mathcal{M}u\|_p + \sum_{i=1}^n \|\mathcal{M}D_i u\|_p \\ &\le A_p\|u\|_p + A_p \sum_{i=1}^n \|D_i u\|_p \le c < \infty \end{aligned}$$

for every $k = 1, 2, \dots$ Hence (v_k) is a bounded sequence in $W^{1,p}(\mathbf{R}^n)$ which converges to $\mathcal{M}u$ pointwise. The weak compactness of Sobolev spaces implies $\mathcal{M}u \in W^{1,p}(\mathbf{R}^n)$, v_k converges to $\mathcal{M}u$ weakly in $L^p(\mathbf{R}^n)$ and $D_i v_k$ converges to $D_i\mathcal{M}u$ weakly in $L^p(\mathbf{R}^n)$. Since $|D_i v_k| \le \mathcal{M}D_i u$ almost everywhere by (2.1), the weak convergence implies

$$|D_i\mathcal{M}u| \le \mathcal{M}D_i u, \quad i = 1, 2, \dots, n,$$

almost everywhere in $\mathbf{R}^n$. ∎

2.2. Remarks: (i) If we only want to prove that $\mathcal{M}u \in W^{1,p}(\mathbf{R}^n), 1 < p < \infty$, whenever u belongs to $W^{1,p}(\mathbf{R}^n)$, there is a simple proof based on the characterization of $W^{1,p}(\mathbf{R}^n)$ by integrated difference quotients, see [3, 7.11]. If $f\colon \mathbf{R}^n \to [-\infty,\infty]$ and $h \in \mathbf{R}^n$, $h \neq 0$, we denote

$$f_h\colon \mathbf{R}^n \to [-\infty,\infty],\ f_h(x) = f(x+h). \tag{2.3}$$

The sublinearity of the maximal operator implies $|\mathcal{M}(u_h) - \mathcal{M}u| \le \mathcal{M}(u_h - u)$ and hence

$$\begin{aligned}\|(\mathcal{M}u)_h - \mathcal{M}u\|_p = \|\mathcal{M}(u_h) - \mathcal{M}u\|_p &\le \|\mathcal{M}(u_h - u)\|_p \\ &\le A_p\|u_h - u\|_p \le A_p\|Du\|_p|h|,\end{aligned}$$

from which the claim follows using [3, Lemma 7.24]. Unfortunately, this argument does not seem to give the pointwise inequality (1.5) for the partial derivatives.

(ii) Inequality (1.5) implies

$$|D\mathcal{M}u(x)| \le \mathcal{M}|Du|(x) \tag{2.4}$$

almost every $x \in \mathbf{R}^n$. To see this, let $x \in \mathbf{R}^n$. If $|D\mathcal{M}u(x)| = 0$, then the claim is obvious. Hence we may assume that $|D\mathcal{M}u(x)| \neq 0$. Now $D_h u = Du \cdot h$ for every $h \in \mathbf{R}^n$ with $|h| = 1$, where D_h denotes the derivative to the direction h. We choose $h = D\mathcal{M}u(x)/|D\mathcal{M}u(x)|$ and rotate the coordinates so that h coincides with some of the coordinate directions, we get

$$|D\mathcal{M}u(x)| = |D_h\mathcal{M}u(x)| \le \mathcal{M}D_hu(x) \le \mathcal{M}|Du|(x).$$

Now we may use the Hardy–Littlewood–Wiener theorem together with (2.4) and obtain

$$\|\mathcal{M}u\|_{1,p} = \|\mathcal{M}u\|_p + \|D\mathcal{M}u\|_p \le A_p\|u\|_p + \big\|\mathcal{M}|Du|\big\|_p \le A_p\|u\|_{1,p},$$

where A_p is the constant in (1.2).

(iii) If $u \in W^{1,\infty}(\mathbf{R}^n)$, then a slight modification of our proof shows that $\mathcal{M}u$ belongs to $W^{1,\infty}(\mathbf{R}^n)$. Moreover,

$$\|\mathcal{M}u\|_{1,\infty} = \|\mathcal{M}u\|_\infty + \|D\mathcal{M}u\|_\infty \le \|u\|_\infty + \big\|\mathcal{M}|Du|\big\|_\infty \le \|u\|_{1,\infty}.$$

Recall, that after a redefinition on a set of measure zero $u \in W^{1,\infty}(\mathbf{R}^n)$ is bounded and Lipschitz continuous. If we are not interested in pointwise estimates, there is a simple proof of the fact that the maximal function maps bounded Lipschitz continuous functions into themselves. Indeed, suppose that u is Lipschitz continuous with constant L, that is

$$|u_h(x) - u(x)| \le L|h|$$

for every $x, h \in \mathbf{R}^n$ where u_h is defined by (2.3). The same argument as in (i) shows that

$$\begin{aligned}|(\mathcal{M}u)_h(x) - \mathcal{M}u(x)| &= |\mathcal{M}(u_h)(x) - \mathcal{M}u(x)| \le \mathcal{M}(u_h - u)(x)\\ &= \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u_h(y) - u(y)|\, dy \le L|h|,\end{aligned}$$

which means that the maximal function is Lipschitz continuous with constant L. Observe, that this proof applies to Hölder continuous functions as well.

(iv) Finally we remark that our method applies to other maximal and maximal singular integral operators as well.

3. A capacity inequality

We show that a weak type inequality for the Sobolev capacity follows immediately from our Theorem 1.4. The standard proof depends on some extension properties of Sobolev functions, see [2]. Let $1 < p < \infty$. The Sobolev p-capacity of the set $E \subset \mathbf{R}^n$ is defined by

$$\mathrm{C}_p(E) = \inf_{u \in \mathcal{A}(E)} \int_{\mathbf{R}^n} \left(|u|^p + |Du|^p\right) dx,$$

where

$$\mathcal{A}(E) = \left\{u \in W^{1,p}(\mathbf{R}^n) : u \ge 1 \text{ on a neighbourhood of } E\right\}.$$

If $\mathcal{A}(E) = \emptyset$, we set $\mathrm{C}_p(E) = \infty$. The Sobolev p-capacity is a monotone and a countably subadditive set function [2]. Let $u \in W^{1,p}(\mathbf{R}^n)$, suppose that $\lambda > 0$ and denote

$$E_\lambda = \{x \in \mathbf{R}^n \colon \mathcal{M}u(x) > \lambda\}.$$

Then E_λ is open and $\mathcal{M}u/\lambda$ is admissible for E_λ. Using (2.4) we get

$$\begin{aligned} \mathrm{C}_p(E_\lambda) &\leq \frac{1}{\lambda^p}\int_{\mathbf{R}^n}\left(|\mathcal{M}u|^p+|D\mathcal{M}u|^p\right)dx \\ &\leq \frac{A_p^p}{\lambda^p}\int_{\mathbf{R}^n}\left(|u|^p+|Du|^p\right)dx \\ &\leq \frac{A_p^p}{\lambda^p}\|u\|_{1,p}^p. \end{aligned} \tag{3.1}$$

This inequality can be used in studying the pointwise behaviour of Sobolev functions by the standard methods, see [2], but we shall use it to prove that the Hardy–Littlewood maximal function of a Sobolev function is quasicontinuous.

4. Quasicontinuity

First we recall some terminology. A property holds p-quasieverywhere if it holds outside a set of the Sobolev p-capacity zero. A function u is p -quasicontinuous in $\mathbf{R}^n$ if for every $\varepsilon > 0$ there is a set F such that $\mathrm{C}_p(F) < \varepsilon$ and the restriction of u to $\mathbf{R}^n \setminus F$ is continuous and finite. It is well known that each Sobolev function has a quasicontinuous representative, see [2]. To be more precise, for each $u \in W^{1,p}(\mathbf{R}^n)$ there is a p-quasicontinuous function $v \in W^{1,p}(\mathbf{R}^n)$ such that $v = u$ a.e. in $\mathbf{R}^n$. Moreover, this representative is unique in the following sense: If v and w are p-quasicontinuous and $v = w$ a.e., then $w = u$ p-quasieverywhere in $\mathbf{R}^n$.

4.1. THEOREM: *If $u \in W^{1,p}(\mathbf{R}^n)$, $1 < p < \infty$, then $\mathcal{M}u$ is p-quasicontinuous.*

Proof: We begin with showing that if $u \in C(\mathbf{R}^n)\cap L^p(\mathbf{R}^n)$, then $\mathcal{M}u \in C(\mathbf{R}^n)\cap L^p(\mathbf{R}^n)$. Indeed, if $x, h \in \mathbf{R}^n$ and $\varepsilon > 0$, then there is $r_\varepsilon < \infty$ such that

$$\begin{aligned} \frac{1}{|B(x,r)|}\int_{B(x,r)}|u_h(y)-u(y)|\,dy &\leq \left(\frac{1}{|B(x,r)|}\int_{B(x,r)}|u_h(y)-u(y)|^p\,dy\right)^{1/p} \\ &\leq \frac{\|u_h-u\|_p}{|B(x,r)|^p} \leq \frac{2\|u\|_p}{|B(x,r)|^p} < \varepsilon \end{aligned}$$

whenever $r > r_\varepsilon$. On the other hand, if $0 < r \leq r_\varepsilon$, then there is $\delta > 0$ such that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)}|u_h(y)-u(y)|\,dy \leq \sup_{B(x,r_\varepsilon)}|u_h-u| < \varepsilon$$

whenever $|h| < \delta$. Thus $\mathcal{M}(u_h-u)(x) \leq \varepsilon$ for $|h| < \delta$ and

$$|(\mathcal{M}u)_h(x)-\mathcal{M}u(x)| \leq \mathcal{M}(u_h-u)(x) \leq \varepsilon$$

whenever $|h| < \delta$. This shows that $\mathcal{M}u$ is continuous at x.

Suppose then that $u \in W^{1,p}(\mathbf{R}^n)$ and let (φ_i) be a sequence of functions $\varphi_i \in C_0^\infty(\mathbf{R}^n)$, $i = 1, 2, \ldots$, so that $\varphi_i \to u$ in $W^{1,p}(\mathbf{R}^n)$. By the weak type inequality (3.1) there is a set F with $\mathrm{C}_p(F) = 0$ so that $\mathcal{M}u$ is finite in $\mathbf{R}^n \setminus F$. We choose a subsequence, which is denoted by (φ_i), such that

$$\|\varphi_i - u\|_{1,p}^p < (4^i A_p)^{-p}.$$

Set $E_i = \{x \in \mathbf{R}^n \setminus F\colon \mathcal{M}(\varphi_i - u)(x) > 2^{-i}\}$, $i = 1, 2, \ldots$ Then using inequality (3.1) we get

$$\mathrm{C}_p(E_i) \leq 2^{ip} A_p^p \|\varphi_i - u\|_{1,p}^p \leq 2^{-ip}.$$

If $F_j = \bigcup_{i=j}^\infty E_i$, then by subadditivity

$$\mathrm{C}_p(F_j) \leq \sum_{i=j}^\infty 2^{-ip} < \infty$$

and hence $\lim_{j\to\infty} \mathrm{C}_p(F_j) = 0$. Moreover, for $x \in \mathbf{R}^n \setminus F_j$ we have

$$|\mathcal{M}\varphi_i(x) - \mathcal{M}u(x)| \leq \mathcal{M}(\varphi_i - u)(x) \leq 2^{-i}$$

whenever $i \geq j$, which shows that the convergence is uniform in $\mathbf{R}^n \setminus F_j$. As a uniform limit of continuous functions $\mathcal{M}u$ is continuous in $\mathbf{R}^n \setminus F_j$. This implies that $\mathcal{M}u$ is p-quasicontinuous. ∎

4.2. Remark: If $p > n$, then every non-empty set has a positive p–capacity and hence the maximal function of a function $u \in W^{1,p}(\mathbf{R}^n)$ is continuous. In fact, by the Sobolev imbedding theorem [3, Theorem 7.17] it is Hölder continuous,

$$|(\mathcal{M}u)_h(x) - \mathcal{M}u(x)| \leq c|h|^{1-n/p}$$

for every $x, h \in \mathbf{R}^n$. If $p = \infty$, the maximal function is Lipschitz continuous.

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